

SMOOTHING OF WEIGHTS IN THE BERNSTEIN APPROXIMATION PROBLEM

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ABSTRACT. In 1924 S. Bernstein [2] asked for conditions on a uniformly bounded on \mathbb{R} Borel function (weight) $w : \mathbb{R} \rightarrow [0, +\infty)$ which imply the denseness of algebraic polynomials \mathcal{P} in the seminormed space C_w^0 defined as the linear set $\{f \in C(\mathbb{R}) \mid w(x)f(x) \rightarrow 0 \text{ as } |x| \rightarrow +\infty\}$ equipped with the seminorm $\|f\|_w := \sup_{x \in \mathbb{R}} w(x)|f(x)|$. In 1998 A. Borichev and M. Sodin [4] completely solved this problem for all those weights w for which \mathcal{P} is dense in C_w^0 but there exists a positive integer $n = n(w)$ such that \mathcal{P} is not dense in $C_{(1+x^2)^n w}^0$. In the present paper we establish that if \mathcal{P} is dense in $C_{(1+x^2)^n w}^0$ for all $n \geq 0$ then for arbitrary $\varepsilon > 0$ there exists a weight $W_\varepsilon \in C^\infty(\mathbb{R})$ such that \mathcal{P} is dense in $C_{(1+x^2)^n W_\varepsilon}^0$ for every $n \geq 0$ and $W_\varepsilon(x) \geq w(x) + e^{-\varepsilon|x|}$ for all $x \in \mathbb{R}$.

1. INTRODUCTION

Let $C(\mathbb{R})$ be the linear space of all continuous real-valued functions on \mathbb{R} , $\mathcal{W}(\mathbb{R})$ the set of all uniformly bounded on \mathbb{R} Borel functions $w : \mathbb{R} \rightarrow \mathbb{R}^+ := [0, +\infty)$ which have an unbounded support $S_w := \{x \in \mathbb{R} \mid w(x) > 0\}$ and satisfy $|x|^n w(x) \rightarrow 0$ as $|x| \rightarrow \infty$ for all $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$. Denote by \mathcal{P} the set of all algebraic polynomials with real coefficients.

For $w \in \mathcal{W}(\mathbb{R})$ the seminormed space $C_w^0(\mathbb{R})$ consists of the linear set of all $f \in C(\mathbb{R})$ with $\lim_{|x| \rightarrow +\infty} w(x)f(x) = 0$ and the semi-norm $\|\cdot\|_w$, where $\|f\|_w := \sup_{x \in \mathbb{R}} w(x)|f(x)|$.

We recall the definition of the so-called *upper Baire function* M_F of $F : \mathbb{R} \rightarrow \mathbb{R}$ as $M_F(x) := \lim_{\delta \downarrow 0} \sup_{y \in (x-\delta, x+\delta)} F(y)$ (see [12, p. 129]). If F is locally bounded from above, then M_F is an upper semi-continuous function and $F(x) \leq M_F(x)$, $x \in \mathbb{R}$. It is easy to verify that for arbitrary $-\infty < A < B < +\infty$, $w \in \mathcal{W}(\mathbb{R})$ and $f \in C(\mathbb{R})$ we have

$$\sup_{x \in (A, B)} w(x)|f(x)| = \sup_{x \in (A, B)} M_w(x)|f(x)|.$$

This means that the seminormed spaces $C_w^0(\mathbb{R})$ and $C_{M_w}^0(\mathbb{R})$ coincide identically and, in particular, \mathcal{P} is dense in $C_w^0(\mathbb{R})$ iff it is dense in $C_{M_w}^0(\mathbb{R})$. Thus, it is possible to assume everywhere below that $w \in \mathcal{W}^*(\mathbb{R})$ where $\mathcal{W}^*(\mathbb{R})$ denotes the family of all those $w \in \mathcal{W}(\mathbb{R})$ which are upper semi-continuous on \mathbb{R} , i.e., $M_w(x) \equiv w(x)$ for all $x \in \mathbb{R}$.

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Introduce

$$\mathcal{W}^{\text{dens}}(\mathbb{R}) := \{w \in \mathcal{W}^*(\mathbb{R}) \mid \mathcal{P} \text{ is dense in } C_w^0(\mathbb{R})\}. \quad (1.1)$$

In 1924 S. Bernstein [2] asked for conditions on $w \in \mathcal{W}^*(\mathbb{R})$ to be in $\mathcal{W}^{\text{dens}}(\mathbb{R})$. This problem is known as *Bernstein's approximation problem*. Various results towards a final solution of Bernstein's approximation problem have been obtained independently by L. Carleson [6](1951), H. Pollard [13](1953), S. N. Mergelyan [11](1958) and L. de Branges [3](1959) (see also the surveys of P. Koosis [8], A. Poltoratski [14] and M. Sodin [15]).

The solution of Bernstein's problem given by L. de Branges [3] in 1959 was slightly improved in 1996 by M. Sodin and P. Yuditskii [16] and attained the following form.

Let f be an entire function, Λ_f be the set of all its zeros, $0 \leq r, \rho < \infty$ and $\sigma_f(\rho) := \overline{\lim}_{r \rightarrow \infty} r^{-\rho} \log M_f(r)$, where $M_f(r) := \sup_{|z|=r} |f(z)|$. We say that f is of *minimal exponential type* if $\sigma_f(1) = 0$. Denote by $\mathcal{E}_0(\mathbb{R})$ the family of all entire functions f of minimal exponential type which are real on the real axis (in short: real) and have only real simple zeros.

Theorem A (L. de Branges, 1959 [3]). *Let $w \in \mathcal{W}^*(\mathbb{R})$. Then \mathcal{P} is not dense in $C_w^0(\mathbb{R})$ if and only if there exists an entire function $B \in \mathcal{E}_0(\mathbb{R})$ such that $\Lambda_B \subset S_w = \{x \in \mathbb{R} \mid w(x) > 0\}$ and*

$$\sum_{\lambda \in \Lambda_B} \frac{1}{w(\lambda)|B'(\lambda)|} < +\infty.$$

In 1958 S. Mergelyan [11] proved that if algebraic polynomials are dense in $C_w^0(\mathbb{R})$ but are not dense in $C_{(1+x^2)^n w}^0(\mathbb{R})$ for some positive integer n , then w has countable support and the number of points in the set $\{x \in \mathbb{R} \mid w(x) > 0, |x| < R\}$ is $o(R)$ as $R \rightarrow +\infty$. Motivated by this result, A. Borichev and M. Sodin in 1998 [4] divided Bernstein's approximation problem into two parts.

Definition 1. Let $w \in \mathcal{W}^*(\mathbb{R})$. It is said that algebraic polynomials \mathcal{P} are *regularly dense* in $C_w^0(\mathbb{R})$ if they are dense in $C_{(1+x^2)^n w}^0(\mathbb{R})$ for all $n \in \mathbb{N}_0$.

Algebraic polynomials \mathcal{P} are called to be *singularly dense* in $C_w^0(\mathbb{R})$ if they are dense in $C_w^0(\mathbb{R})$ but not in $C_{(1+x^2)^n w}^0(\mathbb{R})$ for a certain $n \in \mathbb{N} := \{1, 2, \dots\}$.

Similarly to (1.1), we denote

$$\begin{aligned} \mathcal{W}^{\text{reg}}(\mathbb{R}) &:= \{w \in \mathcal{W}^*(\mathbb{R}) \mid \mathcal{P} \text{ is regularly dense in } C_w^0(\mathbb{R})\}, \\ \mathcal{W}^{\text{sing}}(\mathbb{R}) &:= \{w \in \mathcal{W}^*(\mathbb{R}) \mid \mathcal{P} \text{ is singularly dense in } C_w^0(\mathbb{R})\}. \end{aligned}$$

It is obvious that $\mathcal{W}^{\text{reg}}(\mathbb{R})$ and $\mathcal{W}^{\text{sing}}(\mathbb{R})$ are two non-intersecting classes of weights and

$$\mathcal{W}^{\text{dens}}(\mathbb{R}) = \mathcal{W}^{\text{reg}}(\mathbb{R}) \sqcup \mathcal{W}^{\text{sing}}(\mathbb{R}),$$

where the symbol \sqcup denotes the union of two non-intersecting sets.

Thus, the finding of conditions on a given weight $w \in \mathcal{W}^*(\mathbb{R})$ to be in $\mathcal{W}^{\text{reg}}(\mathbb{R})$ or in $\mathcal{W}^{\text{sing}}(\mathbb{R})$ divides Bernstein's approximation problem into two independent parts: regular and singular, respectively. A complete solution of the singular part was given by A. Borichev and M. Sodin [4] in 1998.

Theorem B. *Let $w \in \mathcal{W}^*(\mathbb{R})$. Algebraic polynomials \mathcal{P} are singularly dense in $C_w^0(\mathbb{R})$ if and only if w is discrete and there exist an entire function $E \in \mathcal{E}_0(\mathbb{R})$ and a nonnegative integer n such that*

$$w(x) = \sum_{\lambda \in \Lambda_E} w(\lambda) \chi_\lambda(x), \quad x \in \mathbb{R}, \quad \chi_\lambda(x) := \begin{cases} 0, & \text{if } x \neq \lambda, \\ 1, & \text{if } x = \lambda, \end{cases}$$

$$\sum_{\lambda \in \Lambda_E} \frac{1}{(1 + \lambda^2)^k w(\lambda) |E'(\lambda)|} \begin{cases} < +\infty, & \text{if } k = n + 1, \\ = +\infty, & \text{if } k = n, \end{cases}$$

and

$$\sum_{\lambda \in \Lambda_F} \frac{1}{w(\lambda) |F'(\lambda)|} = +\infty$$

for arbitrary transcendental entire functions F of minimal exponential type such that $\Lambda_F \subset \Lambda_E$ and E/F is transcendental.

The regular part of Bernstein's approximation problem is still open but the following important result holds.

Theorem C (M. Sodin, 1996 [15]). *If $w \in \mathcal{W}^{\text{reg}}(\mathbb{R})$, then $w(x) + e^{-\delta|x|} \in \mathcal{W}^{\text{reg}}(\mathbb{R})$ for every $\delta > 0$.*

The following statement about perturbations of zeros of an entire function was proved in [1, Lemma 5, p. 237] (2005).

Lemma A. *For an arbitrary entire function $B \in \mathcal{E}_0(\mathbb{R})$ with zeros $\Lambda_B = \{b_n\}_{n \geq 1}$ there exists a constant $C > 0$ and a sequence of real positive numbers $\{\delta_n\}_{n \geq 1}$ such that for any sequence of real numbers $\{d_n\}_{n \geq 1}$ satisfying*

$$|b_n - d_n| \leq \delta_n, \quad n \geq 1,$$

one can find an entire function $D \in \mathcal{E}_0(\mathbb{R})$ such that $\Lambda_D = \{d_n\}_{n \geq 1}$ and

$$|B'(b_n)| \leq C \cdot |D'(d_n)|, \quad n \geq 1.$$

If the set of real numbers $\{|B'(b_n)|\}_{n \geq 1}$ in Lemma A is bounded from below, then the result of Lemma A can be improved as follows.

Lemma 1. *Let $B \in \mathcal{E}_0(\mathbb{R})$ and Λ_B denote the set of its zeros. Assume that*

$$\sum_{\lambda \in \Lambda_B} \frac{1}{|B'(\lambda)|} < \infty. \quad (1.2)$$

Then, for arbitrary $\delta > 0$ there exist constants $C_\delta = C_\delta(B)$, $\rho_\delta = \rho_\delta(B) > 0$ such that for any set of real numbers $\{d_\lambda\}_{\lambda \in \Lambda_B}$ satisfying

$$|\lambda - d_\lambda| \leq \rho_\delta e^{-\delta|\lambda|}, \quad \lambda \in \Lambda_B, \quad (1.3)$$

one can find an entire function $D \in \mathcal{E}_0(\mathbb{R})$ such that $\Lambda_D = \{d_\lambda\}_{\lambda \in \Lambda_B}$ and

$$|B'(\lambda)| \leq C_\delta |D'(d_\lambda)|, \quad \lambda \in \Lambda_B. \quad (1.4)$$

Lemma 1 is instrumental for the proof of the next statement.

Lemma 2. *Let $\varepsilon > 0$ and $w \in \mathcal{W}^{\text{reg}}(\mathbb{R})$. Then,*

$$w_\varepsilon(x) := \sup_{|t| \leq e^{-\varepsilon|x|}} \left(w(x+t) + e^{-\varepsilon|x+t|} \right) \in \mathcal{W}^{\text{reg}}(\mathbb{R}). \quad (1.5)$$

Proof. Assume that $w_\varepsilon \notin \mathcal{W}^{\text{reg}}(\mathbb{R})$. Then, for some $m \in \mathbb{N}_0$ we have $(1+x^{2m})w_\varepsilon \notin \mathcal{W}^{\text{dens}}(\mathbb{R})$ and by Theorem A there exists an entire function $F \in \mathcal{E}_0(\mathbb{R})$ such that

$$\sum_{\lambda \in \Lambda_F} \frac{1}{(1+\lambda^{2m}) w_\varepsilon(\lambda) |F'(\lambda)|} < \infty. \quad (1.6)$$

It follows from $w_\varepsilon \in \mathcal{W}^*(\mathbb{R})$ that $\sum_{\lambda \in \Lambda_F} 1/|F'(\lambda)| < \infty$ and therefore (1.2) holds for $B = F$.

By Theorem C,

$$\beta_\varepsilon(x) := w(x) + e^{-\varepsilon|x|} \in \mathcal{W}^{\text{reg}}(\mathbb{R}), \quad (1.7)$$

and since this function is upper semi-continuous on \mathbb{R} an application of [7, Theorem 1.2, p. 4] to the supremum in (1.5) yields for each $x \in \mathbb{R}$ the existence of $\theta_\varepsilon(x) \in [-1, 1]$ such that

$$w_\varepsilon(x) = \beta_\varepsilon \left(x + \theta_\varepsilon(x) e^{-\varepsilon|x|} \right), \quad x \in \mathbb{R}.$$

From (1.6) we obtain

$$\sum_{\lambda \in \Lambda_F} \frac{1}{(1+\lambda^{2m}) \beta_\varepsilon(\lambda + \theta_\varepsilon(\lambda) e^{-\varepsilon|\lambda|}) |F'(\lambda)|} < \infty. \quad (1.8)$$

Applying Lemma 1 for $\delta = \varepsilon/2$, we find $T_\varepsilon > 0$ such that $e^{-\varepsilon x/2} \leq \rho_{\varepsilon/2}$, $x \geq T_\varepsilon$, and then we find an entire function $D \in \mathcal{E}_0(\mathbb{R})$ with zeros $\Lambda_D = \{d_\lambda\}_{\lambda \in \Lambda_B}$, where

$$d_\lambda = \lambda, \quad \lambda \in \Lambda_F \cap [-T_\varepsilon, T_\varepsilon], \quad d_\lambda = \lambda + \theta(\lambda) e^{-\varepsilon|\lambda|}, \quad \lambda \in \Lambda_F \setminus [-T_\varepsilon, T_\varepsilon].$$

Hence, in view of (1.4) and (1.8) we have

$$\begin{aligned} \infty &> \sum_{\lambda \in \Lambda_F \setminus [-T_\varepsilon, T_\varepsilon]} \frac{1}{(1+\lambda^{2m}) \beta_\varepsilon(\lambda + \theta_\varepsilon(\lambda) e^{-\varepsilon|\lambda|}) |F'(\lambda)|} \\ &\geq \frac{1}{C_{\varepsilon/2}} \sum_{\lambda \in \Lambda_F \setminus [-T_\varepsilon, T_\varepsilon]} \frac{1+d_\lambda^{2m}}{1+\lambda^{2m}} \frac{1}{(1+d_\lambda^{2m}) \beta_\varepsilon(d_\lambda) |D'(d_\lambda)|} \\ &\geq \frac{1}{2^{2m} C_{\varepsilon/2}} \sum_{\lambda \in \Lambda_F \setminus [-T_\varepsilon, T_\varepsilon]} \frac{1}{(1+d_\lambda^{2m}) \beta_\varepsilon(d_\lambda) |D'(d_\lambda)|}, \end{aligned}$$

from which it follows that

$$\sum_{\lambda \in \Lambda_D} (1+\lambda^{2m})^{-1} \beta_\varepsilon(\lambda)^{-1} |D'(\lambda)|^{-1} < \infty.$$

By Theorem A this means that $(1+x^{2m}) \cdot \beta_\varepsilon \notin \mathcal{W}^{\text{dens}}(\mathbb{R})$ and therefore $\beta_\varepsilon \notin \mathcal{W}^{\text{reg}}(\mathbb{R})$. This contradicts (1.7) and finishes the proof of Lemma 2. \square

We are now ready to prove our main result.

Theorem 1. *For arbitrary $w \in \mathcal{W}^{\text{reg}}(\mathbb{R})$ and $\varepsilon > 0$ there exists $W_\varepsilon \in C^\infty(\mathbb{R})$ such that $W_\varepsilon \in \mathcal{W}^{\text{reg}}(\mathbb{R})$ and $W_\varepsilon(x) \geq w(x) + e^{-\varepsilon|x|}$ for all $x \in \mathbb{R}$.*

Proof. Since the statement of the theorem for $\varepsilon = \varepsilon_0 > 0$ implies its validity for all $\varepsilon \geq \varepsilon_0$, we can assume without loss of generality that $\varepsilon \in (0, 1)$.

Let w_ε be defined as in (1.5), β_ε as in (1.7) and

$$\Omega_\rho(x) := \sup_{|s| \leq \rho e^{-\varepsilon|x|}} \beta_\varepsilon(x+s), \quad x \in \mathbb{R}, \quad \rho \in (0, 1].$$

Since

$$w(x) \leq w(x) + e^{-\varepsilon|x|} = \beta_\varepsilon(x) \leq \Omega_\rho(x) \leq w_\varepsilon(x), \quad x \in \mathbb{R}, \quad (1.9)$$

by Lemma 2,

$$\Omega_\rho \in \mathcal{W}^{\text{reg}}(\mathbb{R}), \quad \rho \in (0, 1].$$

Let us introduce

$$K_\theta(x, t) := \left(\int_{-\theta e^{-x^2}}^{\theta e^{-x^2}} \exp\left(-\frac{\theta^2 e^{-2x^2}}{\theta^2 e^{-2x^2} - s^2}\right) ds \right)^{-1} \exp\left(-\frac{\theta^2 e^{-2x^2}}{\theta^2 e^{-2x^2} - t^2}\right) \chi_{[-\theta e^{-x^2}, \theta e^{-x^2}]}(t),$$

where $x, t \in \mathbb{R}$ and

$$4\theta := e^{-\frac{\varepsilon^2}{4}}.$$

Obviously,

$$\int_{\mathbb{R}} K_\theta(x, t) dt = 1, \quad \theta e^{-x^2} \leq \frac{1}{4} \cdot e^{-\varepsilon|x|}, \quad x \in \mathbb{R}, \quad (1.10)$$

and therefore the weight

$$W_\varepsilon(x) := \int_{-\theta e^{-x^2}}^{\theta e^{-x^2}} K_\theta(x, t) \Omega_{1/2}(x+t) dt = \int_{x-\theta e^{-x^2}}^{x+\theta e^{-x^2}} K_\theta(x, t-x) \Omega_{1/2}(t) dt$$

belongs to $C^\infty(\mathbb{R})$.

Let $x \in \mathbb{R}$ be arbitrary and let $t \in \mathbb{R}$ satisfy $|t| \leq \theta e^{-x^2}$. Then, by (1.10) we have $|t| \leq e^{-\varepsilon|x|}/4$ and the inequalities $e^{1/4} \leq 4/3$ and $0 < \varepsilon < 1$ imply $(3/4)e^{-\varepsilon|x|} \leq e^{-\varepsilon|x+t|} \leq (4/3)e^{-\varepsilon|x|}$. Thus, for every $\rho \in (1/3, 1]$,

$$(3\rho - 1)e^{-\varepsilon|x|}/4 \leq \rho e^{-\varepsilon|x+t|} + t \leq (16\rho + 3)e^{-\varepsilon|x|}/12,$$

and therefore

$$\Omega_{(3\rho-1)/4}(x) \leq \Omega_\rho(x+t) \leq \Omega_{(16\rho+3)/12}(x), \quad \rho \in (1/3, 1), \quad |t| \leq e^{-\varepsilon|x|}/4, \quad x \in \mathbb{R},$$

from which we infer that

$$\beta_\varepsilon(x) \leq \Omega_{1/8}(x) \leq \Omega_{1/2}(x+t) \leq \Omega_{11/12}(x) \leq \Omega_1(x) = w_\varepsilon(x), \quad |t| \leq \theta e^{-x^2}, \quad x \in \mathbb{R}.$$

In view of (1.9) this means that the weight W_ε satisfies

$$w(x) + e^{-\varepsilon|x|} \leq W_\varepsilon(x) \leq w_\varepsilon(x), \quad x \in \mathbb{R}. \quad (1.11)$$

It follows from the right-hand side inequality of (1.11) that $W_\varepsilon \in \mathcal{W}^{\text{reg}}(\mathbb{R})$ and therefore the left-hand side inequality of (1.11) completes the proof. \square

Theorem 1 allows to assume without loss of generality that each weight in the regular part of Bernstein's approximation problem is continuous and positive on the whole real axis. It also allows to apply for this part of the problem the sufficient conditions for the denseness of algebraic polynomials in $C_w^0(\mathbb{R})$ obtained earlier under this assumption (see [13, p.869], [11, p.80]). On the other hand, Lemma 2 makes it possible to replace any weight $w \in \mathcal{W}^*(\mathbb{R})$ by the greater step function

$$\begin{aligned}\widehat{w}(x) &= \sum_{n \in \mathbb{Z}} w_n \chi_{[\sigma_n \log(1+|n|), \sigma_n \log(1+|n+1|)]}(x), \quad x \in \mathbb{R}, \\ w_n &:= \sup_{x \in [\sigma_n \log(1+|n|), \sigma_n \log(1+|n+1|)]} w(x), \quad \sigma_n := \text{sign}(n), \quad n \in \mathbb{Z},\end{aligned}$$

such that algebraic polynomials are regularly dense in $C_w^0(\mathbb{R})$ if and only if they are regularly dense in $C_{\widehat{w}}^0(\mathbb{R})$. Here, $\text{sign}(n)$ is equal to 1 if $n > 0$, 0 if $n = 0$ and -1 if $n < 0$.

2. AUXILIARY RESULTS

Lemma 3. *Let the real numbers a, b, x and $\Delta \in (0, 1)$ satisfy*

$$b \in (a - \Delta^2, a + \Delta^2), \quad 0 \notin (a - \Delta, a + \Delta) \quad \text{and} \quad x \notin (a - 2\Delta, a + 2\Delta). \quad (2.1)$$

Then,

$$\left| \left(1 - \frac{x}{a}\right) \left(1 - \frac{x}{b}\right)^{-1} \right| \leq (1 + \Delta)^2.$$

Proof. The conditions (2.1) imply $|a| \geq \Delta$, $b \in (a - \Delta, a + \Delta)$ and therefore $|x - b| \geq \Delta$. Thus, $||b| - |a|| \leq |b - a| \leq \Delta^2$ and $|b| \leq |a| + \Delta^2$, i.e. $|b|/|a| \leq 1 + \Delta^2/|a| \leq 1 + \Delta$. Finally,

$$\begin{aligned}\left| \frac{1 - x/a}{1 - x/b} \right| &= \frac{|b|}{|a|} \frac{|x - a|}{|x - b|} = \frac{|b|}{|a|} \frac{|b - a + (x - b)|}{|x - b|} \\ &\leq \frac{|b|}{|a|} \frac{|b - a| + |x - b|}{|x - b|} \leq (1 + \Delta) \cdot \left(1 + \frac{|b - a|}{|x - b|}\right) \leq (1 + \Delta)^2,\end{aligned}$$

which completes the proof. \square

Lemma 4. *Let $\varepsilon \in (0, 1/(2e))$, $C_\varepsilon \in (0, +\infty)$ and f be an entire function satisfying*

$$|f(z)| \leq C_\varepsilon e^{\varepsilon|z|}, \quad z \in \mathbb{C}. \quad (2.2)$$

Then,

$$|f'(z)|, \quad \left| \frac{f(z)}{z - \lambda} \right| \leq C_\varepsilon e^{\varepsilon|z|}, \quad \lambda \in \Lambda_f, \quad z \in \mathbb{C}.$$

Proof. Cauchy's formula [17, (3), p. 81]

$$f'(z) = \frac{1}{2\pi i} \int_{|z - \zeta| = 1/\varepsilon} \frac{f(\zeta) d\zeta}{(\zeta - z)^2}$$

and (2.2) for any $z \in \mathbb{C}$ yield

$$\begin{aligned}|f'(z)| &\leq \varepsilon \max_{|\zeta - z| = 1/\varepsilon} |f(\zeta)| \leq \varepsilon C_\varepsilon \max_{|\zeta - z| = 1/\varepsilon} e^{\varepsilon|\zeta|} \\ &\leq \varepsilon C_\varepsilon e^{\varepsilon(|z| + 1/\varepsilon)} = \varepsilon e C_\varepsilon e^{\varepsilon|z|} \leq C_\varepsilon e^{\varepsilon|z|}.\end{aligned}$$

For arbitrary $\lambda \in \Lambda_f$ and $z \in \mathbb{C}$ satisfying $|z - \lambda| \geq 1/(2\varepsilon)$ it follows from (2.2) that

$$\left| \frac{f(z)}{z - \lambda} \right| \leq 2\varepsilon C_\varepsilon e^{\varepsilon|z|} \leq C_\varepsilon e^{\varepsilon|z|},$$

which by the maximum modulus principle [17, p. 165] yields

$$\begin{aligned} \left| \frac{f(z)}{z - \lambda} \right| &\leq \max_{|\zeta - \lambda| = 1/(2\varepsilon)} \left| \frac{f(\zeta)}{\zeta - \lambda} \right| = 2\varepsilon \max_{|\zeta - \lambda| = 1/(2\varepsilon)} |f(\zeta)| \\ &\leq 2\varepsilon C_\varepsilon \max_{|\zeta - \lambda| = 1/(2\varepsilon)} e^{\varepsilon|\zeta|} \leq 2\varepsilon C_\varepsilon e^{\varepsilon(|z| + 1/\varepsilon)} \leq 2\varepsilon e C_\varepsilon e^{\varepsilon|z|} \leq C_\varepsilon e^{\varepsilon|z|}, \end{aligned}$$

provided that $|z - \lambda| \leq 1/(2\varepsilon)$. This finishes the proof of Lemma 4. \square

Lemma 5. *Let $\varepsilon \in (0, 1/(2e))$, $C_\varepsilon \in (0, +\infty)$ and B be an entire function from the class $\mathcal{E}_0(\mathbb{R})$ satisfying*

$$(a) \quad |B(z)| \leq C_\varepsilon e^{\varepsilon|z|}, \quad z \in \mathbb{C}, \quad (b) \quad \Theta_B := \sum_{\lambda \in \Lambda_B} \frac{1}{|B'(\lambda)|} < \infty. \quad (2.3)$$

Then, for arbitrary $\lambda \in \Lambda_B$ the inequality

$$\left| \frac{B(x)}{x - \lambda} \right| \geq |B'(\lambda)| / 2 \quad (2.4)$$

holds for every real x satisfying

$$|x - \lambda| \leq \frac{e^{-\varepsilon}}{1 + 2C_\varepsilon \Theta_B} e^{-\varepsilon|\lambda|}. \quad (2.5)$$

Thus,

$$\min_{\mu \in \Lambda_B \setminus \{\lambda\}} |\lambda - \mu| > \frac{e^{-\varepsilon}}{1 + 2C_\varepsilon \Theta_B} e^{-\varepsilon|\lambda|}, \quad \lambda \in \Lambda_B. \quad (2.6)$$

Proof. Let $\lambda \in \Lambda_B$ and

$$B_\lambda(x) := \frac{B(x)}{x - \lambda}.$$

Obviously, $B_\lambda(\lambda) = B'(\lambda)$ and it follows from Lemma 4 that

$$|B'_\lambda(z)| \leq C_\varepsilon e^{\varepsilon|z|}, \quad z \in \mathbb{C}.$$

Furthermore, (2.3)(b) yields

$$|B'(\lambda)| \geq \Theta_B^{-1}.$$

Assume that $x \in [-1, 1]$ and

$$|B_\lambda(x + \lambda) - B_\lambda(\lambda)| > |B_\lambda(\lambda)| / 2.$$

Then,

$$\begin{aligned} \Theta_B^{-1}/2 &\leq |B'(\lambda)| / 2 = |B_\lambda(\lambda)| / 2 < |B_\lambda(x + \lambda) - B_\lambda(\lambda)| \\ &= \left| \int_0^{|x|} B'_\lambda(\lambda + \sigma t) dt \right| \leq C_\varepsilon \int_0^{|x|} e^{\varepsilon|\lambda + \sigma t|} dt \leq C_\varepsilon e^{\varepsilon(1 + |\lambda|)} |x| \\ &< (C_\varepsilon + \Theta_B^{-1}/2) e^{\varepsilon(1 + |\lambda|)} |x|, \end{aligned}$$

where $\sigma = 1$ if $x > 0$ and $\sigma = -1$ if $x < 0$. This means that if

$$|x| \leq \frac{e^{-\varepsilon(1 + |\lambda|)}}{1 + 2C_\varepsilon \Theta_B},$$

then

$$|B_\lambda(x + \lambda) - B'(\lambda)| \leq |B'(\lambda)|/2,$$

and therefore

$$\begin{aligned} |B_\lambda(\lambda + x)| &= |B'(\lambda) + B_\lambda(\lambda + x) - B'(\lambda)| \\ &\geq |B'(\lambda)| - |B_\lambda(\lambda + x) - B'(\lambda)| \geq |B'(\lambda)|/2, \end{aligned}$$

which was to be proved. \square

3. PROOF OF LEMMA 1

3.1. If Lemma 1 is proved for $\delta = \delta_0 > 0$, then for arbitrary $\delta_1 > \delta_0$ it follows from $|\lambda - d_\lambda| \leq \rho_{\delta_0} e^{-\delta_1|\lambda|} \leq \rho_{\delta_0} e^{-\delta_0|\lambda|}$, $\lambda \in \Lambda_B$ that Lemma 1 also holds for $\delta = \delta_1$ with $C_{\delta_1} = C_{\delta_0}$ and $\rho_{\delta_1} = \rho_{\delta_0}$. Therefore, it is sufficient to prove Lemma 1 only for those numbers δ which satisfy

$$0 < \delta < 1/e.$$

3.2. Let B be an entire function satisfying the conditions of Lemma 1. Then, these conditions are met by any translation of B of the form $B_{T_a}(z) := B(z + a)$, $a \in \mathbb{R} \setminus \{0\}$ because $\Lambda_{B_{T_a}} = \Lambda_B - a$, $\Theta_{B_{T_a}} = \Theta_B$ and $B_{T_a} \in \mathcal{E}_0(\mathbb{R})$, where Θ_B denotes the value of the series in (1.2).

We show that if Lemma 1 is proved for the function B then it also holds for any B_{T_a} , $a \in \mathbb{R} \setminus \{0\}$, with constants $\rho_\delta(B_{T_a}) = e^{-\delta|a|}\rho_\delta(B)$ and $C_\delta(B_{T_a}) = C_\delta(B)$.

Let $\delta > 0$, a be an arbitrary nonzero real number and $E := B_{T_a}$. If $\{e_\lambda\}_{\lambda \in \Lambda_E}$ is any collection of real numbers satisfying $|\lambda - e_\lambda| \leq \rho_\delta(E) \exp(-\delta|\lambda|)$, $\lambda \in \Lambda_E$, then in view of $\Lambda_E = \Lambda_B - a$ we have

$$|\lambda - a - e_{\lambda-a}| \leq \rho_\delta(E) e^{-\delta|\lambda-a|} \leq e^{\delta|a|} \rho_\delta(E) e^{-\delta|\lambda|} = \rho_\delta(B) e^{-\delta|\lambda|}, \quad \lambda \in \Lambda_B,$$

and therefore the numbers $d_\lambda := e_{\lambda-a} + a$, $\lambda \in \Lambda_B$, satisfy condition (1.3). Thus, there exists an entire function $D \in \mathcal{E}_0(\mathbb{R})$ such that $\Lambda_D = \{d_\lambda\}_{\lambda \in \Lambda_B}$ and $|B'(\lambda)| \leq C_\delta(B) |D'(d_\lambda)|$, $\lambda \in \Lambda_B$. Then, for the function $G(z) := D(z + a)$ we have $G \in \mathcal{E}_0(\mathbb{R})$, $\Lambda_G = \Lambda_D - a = \{d_\lambda - a\}_{\lambda \in \Lambda_B} = \{d_{\lambda+a} - a\}_{\lambda \in \Lambda_E} = \{e_\lambda\}_{\lambda \in \Lambda_E}$ and $|E'(\lambda)| = |B'(a + \lambda)| \leq C_\delta(B) |D'(d_{a+\lambda})| = C_\delta(B) |G'(d_{a+\lambda} - a)| = C_\delta(B) |G'(e_\lambda)|$, $\lambda \in \Lambda_E$. This implies the validity of Lemma 1 for B_{T_a} , as claimed.

We conclude that to prove Lemma 1 for all translations B_{T_a} , $a \in \mathbb{R}$, of the entire function B it is sufficient to prove it for at least one of them. We specify the translation of B by choosing an $a \in \mathbb{R} \setminus \Lambda_B$ such that $\min_{\lambda \in \Lambda_B, \lambda > a} (\lambda - a) = \min_{\lambda \in \Lambda_B, \lambda < a} (a - \lambda)$ if Λ_B is unbounded in both directions, $a > 1 + \max \Lambda_B$ if Λ_B is bounded from above and $a < -1 + \min \Lambda_B$ if Λ_B is bounded from below. Considering such B_{T_a} as the initial function B in Lemma 1, we can therefore assume that the set Λ_B of all zeros of B in Lemma 1 obeys the following additional properties:

- (a) $0 \notin \Lambda_B$;
 - (b) $\min_{\lambda \in \Lambda_B, \lambda > 0} |\lambda| = \min_{\lambda \in \Lambda_B, \lambda < 0} |\lambda|$ if $\sup \Lambda_B = +\infty$ and $\inf \Lambda_B = -\infty$;
 - (c) $\min \Lambda_B > 1$ if $\inf \Lambda_B > -\infty$;
 - (d) $\max \Lambda_B < -1$ if $\sup \Lambda_B < +\infty$.
- $$(3.1)$$

Observe that (3.1)(b) means the existence of two neighboring zeros $\lambda_1, \lambda_2 \in \Lambda_B$ of B (i.e., $\lambda_1 < \lambda_2$, $(\lambda_1, \lambda_2) \cap \Lambda_B = \emptyset$) such that $\lambda_1 = -\lambda_2$.

3.3. Denote by Θ_B the value of the series in (1.2) and let

$$\varepsilon := \delta/2 \in (0, 1/(2e)), \quad \rho_\delta := \left(\frac{e^{-\varepsilon}}{4 + 8C_\varepsilon \Theta_B} \right)^2 \in (0, 1/16), \quad (3.2)$$

where

$$C_\varepsilon := \sup_{z \in \mathbb{C}} e^{-\varepsilon|z|} |B(z)| < \infty.$$

Then, for the function B the conditions of Lemma 5 are fulfilled and (2.5) implies that

$$[\lambda_1 - 2\Delta_{\lambda_1}, \lambda_1 + 2\Delta_{\lambda_1}] \cap [\lambda_2 - 2\Delta_{\lambda_2}, \lambda_2 + 2\Delta_{\lambda_2}] = \emptyset, \quad \lambda_1, \lambda_2 \in \Lambda_B, \quad \lambda_1 \neq \lambda_2, \quad (3.3)$$

where

$$\Delta_\lambda := \sqrt{\rho_\delta} e^{-\varepsilon|\lambda|} \in (0, 1/4), \quad \lambda \in \Lambda_B. \quad (3.4)$$

Actually, assume that there exist $\lambda_1, \lambda_2 \in \Lambda_B$ such that $\lambda_1 < \lambda_2$, $(\lambda_1, \lambda_2) \cap \Lambda_B = \emptyset$ and

$$\lambda_1 + 2\Delta_{\lambda_1} \geq \lambda_2 - 2\Delta_{\lambda_2}. \quad (3.5)$$

By virtue of (2.6),

$$\lambda_1 < \lambda_2 - 4\Delta_{\lambda_2}, \quad \lambda_1 + 4\Delta_{\lambda_1} < \lambda_2, \quad (3.6)$$

and therefore

$$\lambda_2 - \lambda_1 > 2\Delta_{\lambda_1} + 2\Delta_{\lambda_2},$$

which contradicts (3.5) and proves (3.3).

Introduce the following neighborhood of Λ_B :

$$\Lambda_B^\Delta := \bigsqcup_{\lambda \in \Lambda_B} [\lambda - 2\Delta_\lambda, \lambda + 2\Delta_\lambda]. \quad (3.7)$$

We now prove that for any two neighboring zeros $\lambda_1 < \lambda_2$ of B the midpoint of the interval $[\lambda_1, \lambda_2]$ does not belong to Λ_B^Δ . In fact, it follows from $\lambda_1, \lambda_2 \in \Lambda_B$, $\lambda_1 < \lambda_2$, $(\lambda_1, \lambda_2) \cap \Lambda_B = \emptyset$ and (3.6) that

$$\frac{\lambda_1 + \lambda_2}{2} < \lambda_2 - 2\Delta_{\lambda_2}, \quad \lambda_1 + 2\Delta_{\lambda_1} < \frac{\lambda_1 + \lambda_2}{2},$$

which proves

$$\lambda_1 < \lambda_2, \quad \lambda_1, \lambda_2 \in \Lambda_B, \quad (\lambda_1, \lambda_2) \cap \Lambda_B = \emptyset \Rightarrow \frac{\lambda_1 + \lambda_2}{2} \notin \Lambda_B^\Delta. \quad (3.8)$$

Together with (3.1) this property means that

$$0 \notin \Lambda_B^\Delta. \quad (3.9)$$

Actually, if Λ_B is unbounded in both directions, then according to (3.1)(b) the origin is the midpoint of a segment joining two neighboring zeros of B which have opposite signs. It follows from (3.8) that (3.9) holds. In the case when Λ_B is bounded from one side the distance $\min_{\lambda \in \Lambda_B} |\lambda|$ between 0 and Λ_B is greater than 1, by virtue of (3.1)(c), (d). But in view of (3.4), $2\Delta_\lambda < 1/2$ and therefore (3.9) follows readily from (3.7).

3.4. If $\{d_\lambda\}_{\lambda \in \Lambda_B}$ are arbitrary numbers satisfying (1.3), it follows from (1.3), (3.2) and (3.4) that

$$d_\lambda \in [\lambda - \Delta_\lambda^2, \lambda + \Delta_\lambda^2] \subset [\lambda - \Delta_\lambda, \lambda + \Delta_\lambda], \quad \lambda \in \Lambda_B, \quad (3.10)$$

and in view of (3.3),

$$d_{\lambda_0} \notin [\lambda - 2\Delta_\lambda, \lambda + 2\Delta_\lambda], \quad \lambda_0, \lambda \in \Lambda_B, \quad \lambda_0 \neq \lambda. \quad (3.11)$$

It is worth remembering that according to the Lindelöf theorem [10, Th. 15, p. 28] a set $\Lambda \subset \mathbb{R} \setminus \{0\}$ is the set of all zeros of some entire function from the class $\mathcal{E}_0(\mathbb{R})$ if and only if there exists a finite limit of $\delta_\Lambda(R)$ and $n_\Lambda(R)/R \rightarrow 0$ as $R \rightarrow +\infty$. Here,

$$\delta_\Lambda(R) := \sum_{\lambda \in \Lambda \cap (-R, R)} 1/\lambda, \quad n_\Lambda(R) := \text{card} \{ \lambda \in \Lambda \mid |\lambda| < R \}, \quad R > 0,$$

and $\text{card } A \in \mathbb{N}_0 \cup \{+\infty\}$ denotes the number of elements in a set A . Then, all functions $f \in \mathcal{E}_0(\mathbb{R})$ satisfying $\Lambda_f = \Lambda$ are given by the following formula:

$$f(z) = A \lim_{R \rightarrow \infty} \prod_{\lambda \in \Lambda \cap (-R, R)} (1 - z/\lambda), \quad A \in \mathbb{R} \setminus \{0\}, \quad z \in \mathbb{C},$$

where $f(0) = A \neq 0$. Thus,

$$B(z) = B(0) \lim_{R \rightarrow \infty} \prod_{\lambda \in \Lambda_B \cap (-R, R)} (1 - z/\lambda), \quad z \in \mathbb{C}. \quad (3.12)$$

and it follows from $\lim_{R \rightarrow +\infty} n_B(R)/R = 0$ that $\sum_{\lambda \in \Lambda_B} 1/\lambda^2 < \infty$.

Denote $\Lambda_D := \{d_\lambda\}_{\lambda \in \Lambda_B}$. Since $\Lambda_B = \{\lambda\}_{\lambda \in \Lambda_B}$ satisfies the conditions of Lindelöf's theorem, they are also met by the set Λ_D because

$$|d_\lambda - \lambda| \leq \frac{\rho_\delta}{\delta^2 \lambda^2}, \quad \lambda \in \Lambda_B,$$

by virtue of (1.3) and the inequality

$$\exp(-x) \leq 1/x^2, \quad x > 0. \quad (3.13)$$

Therefore, Λ_D is the set of all zeros of the entire function

$$D(z) := \lim_{R \rightarrow \infty} \prod_{\substack{\lambda \in \Lambda_B \\ d_\lambda \in (-R, R)}} (1 - z/d_\lambda), \quad z \in \mathbb{C}, \quad (3.14)$$

which belongs to the class $\mathcal{E}_0(\mathbb{R})$.

Let m denote the Lebesgue measure on \mathbb{R} . Then it follows from (3.4), (3.7) and (3.13) that

$$m(\Lambda_B^\Delta) \leq 4\sqrt{\rho_\delta} \sum_{\lambda \in \Lambda_B} e^{-\varepsilon|\lambda|} \leq 4\varepsilon^{-2}\sqrt{\rho_\delta} \sum_{\lambda \in \Lambda_B} 1/\lambda^2 < \infty.$$

Hence, the set

$$\mathbb{R}_B^+ := [0, +\infty) \setminus (\Lambda_B^\Delta \cup -\Lambda_B^\Delta)$$

is unbounded and in view of (3.10), (3.3) and (3.7) we have

$$\{\lambda \mid \lambda \in \Lambda_B \cap (-R, R)\} = \{\lambda \mid \lambda \in \Lambda_B, \quad d_\lambda \in (-R, R)\}, \quad R \in \mathbb{R}_B^+. \quad (3.15)$$

3.5. Let us estimate $|B'(\lambda_0)|/|D'(d_{\lambda_0})|$ for arbitrary $\lambda_0 \in \Lambda_B$. It follows from (2.4), (2.5), (3.2) and (3.4) that

$$\left| \frac{B(x)}{x - \lambda} \right| \geq |B'(\lambda)|/2, \quad x \in [\lambda - 4\Delta_\lambda, \lambda + 4\Delta_\lambda], \quad \lambda \in \Lambda_B,$$

and therefore, by (3.10), we have

$$|B'(\lambda_0)| \leq \frac{2}{|\lambda_0|} \frac{|B(d_{\lambda_0})|}{\left| 1 - \frac{d_{\lambda_0}}{\lambda_0} \right|}.$$

Then, by (3.12), (3.14) and (3.15),

$$\begin{aligned} \frac{|B'(\lambda_0)|}{|D'(d_{\lambda_0})|} &\leq \frac{2}{|\lambda_0|} \frac{|B(d_{\lambda_0})|}{\left| 1 - \frac{d_{\lambda_0}}{\lambda_0} \right| |D'(d_{\lambda_0})|} \\ &= \frac{2|d_{\lambda_0}||B(0)|}{|\lambda_0|} \lim_{\substack{R \rightarrow +\infty \\ R \in \mathbb{R}_B^+}} \prod_{\substack{\lambda \in \Lambda_B \cap (-R, R) \\ \lambda \neq \lambda_0}} \left| \frac{1 - d_{\lambda_0}/\lambda}{1 - d_{\lambda_0}/d_\lambda} \right|. \end{aligned} \quad (3.16)$$

The relations (3.7) and (3.9) imply that $0 \notin [\lambda - 2\Delta_\lambda, \lambda + 2\Delta_\lambda]$ and therefore $|\lambda| \leq 2\Delta_\lambda$, which together with the consequence $|d_\lambda| \leq |\lambda| + \Delta_\lambda^2$ of (3.10) yields in view of (3.4) $|d_\lambda/\lambda| \leq 1 + \Delta_\lambda^2/|\lambda| \leq 1 + \Delta_\lambda/2 \leq 2$ for every $\lambda \in \Lambda_B$. Thus, in (3.16) we have $|d_{\lambda_0}|/|\lambda_0| \leq 2$.

Setting in Lemma 3, $x = d_{\lambda_0}$, $a = \lambda$, $b = d_\lambda$ and $\Delta = \Delta_\lambda$ with λ_0 and λ taken from (3.16), we obtain the validity of the conditions (2.1),

$$d_\lambda \in (\lambda - \Delta_\lambda^2, \lambda + \Delta_\lambda^2), \quad 0 \notin (\lambda - \Delta_\lambda, \lambda + \Delta_\lambda), \quad d_{\lambda_0} \notin (\lambda - 2\Delta_\lambda, \lambda + 2\Delta_\lambda),$$

as a consequence of (3.10), (3.7), (3.9), and (3.11). Hence, the factors in (3.16) satisfy

$$\left| (1 - d_{\lambda_0}/\lambda)(1 - d_{\lambda_0}/d_\lambda)^{-1} \right| \leq (1 + \Delta_\lambda)^2 = \left(1 + \sqrt{\rho_\delta} e^{-\varepsilon|\lambda|} \right)^2,$$

by virtue of (3.4). It follows therefore from (3.16) that

$$\frac{|B'(\lambda_0)|}{|D'(d_{\lambda_0})|} \leq C_\delta := 4|B(0)| \prod_{\lambda \in \Lambda_B} \left(1 + \sqrt{\rho_\delta} e^{-\varepsilon|\lambda|} \right)^2 < \infty, \quad \lambda_0 \in \Lambda_B,$$

where the product above is finite in view of (3.13), (3.1)(a) and $\sum_{\lambda \in \Lambda_B} 1/\lambda^2 < \infty$. Lemma 1 is proved.

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